Some theoretical results of synchronization of a linearly coupled dynamical system with random perturbation on a network

2009 J. Phys. A: Math. Theor. 42 065101

(http://iopscience.iop.org/1751-8121/42/6/065101)

J. Phys. A: Math. Theor. **42** (2009) 065101 (14pp) doi:10.1088/1751-8113/42/6/065101

Some theoretical results of synchronization of a linearly coupled dynamical system with random perturbation on a network

Yong Liu and Fengxia Yang

LMAM, School of Mathematical Sciences, Peking University, Beijing 100871, People's Republic of China

E-mail: liuyong@math.pku.edu.cn and yangfx@math.pku.edu.cn

Received 22 July 2008, in final form 1 December 2008
Published 14 January 2009
Online at stacks.iop.org/JPhysA/42/065101

Abstract
This paper attempts to investigate synchronization of a coupled dynamical system with random noise on a network (graph). Some mathematical results are obtained in this paper, first, a synchronized threshold is given for a linearly coupled dynamical system perturbed by the random noise; second, when the linearly coupled dynamical system with random noise reaches synchronization, the long time behavior of identical orbit on each node is discussed; finally, some interesting comparison between the synchronization of the coupled gradient system and its small random perturbed system is considered.

PACS numbers: 05.45.Xt, 02.50.Fz, 05.40.Jc

1. Introduction

Synchronization is a universal phenomenon in a variety of fields of engineering and sciences. For example, semiconductor lasers and fireflies to fire in unison (see [14]). Recently, an increasing number of articles have been devoted to investigate synchronization phenomena in complex networks of linearly coupled identical dynamical systems (see [3, 6, 10–12, 17–20]). The basic mathematical model in most of these articles is studied as follows (see [6, 10, 17]).

The dynamical network consists of N identical linearly and diffusively coupled nodes, with each node being a d-dimensional dynamical system. The state equations of the network are

$$\dot{\mathbf{x}}_i = f(\mathbf{x}_i) + c\sum_{\substack{j=1\\ j\neq i}}^{N} a_{ij}\Gamma(\mathbf{x}_j - \mathbf{x}_i), \qquad i = 1, 2, \ldots, N, \tag{1.1}$$

where $\mathbf{x}_i = (x_{i_1}, x_{i_2}, \ldots, x_{i_d})^T \in \mathbb{R}^d$ are the state variables of node i, the constant $c > 0$ represents the coupling strength, and, for simplicity, $\Gamma = \mathrm{diag}(\underbrace{1, 1, \ldots, 1}_{q}, 0, 0, \ldots, 0) \in \mathbb{R}^{d\times d}$, $1 \leqslant q \leqslant d$ is a diagonal matrix linking the corresponding components of coupled nodes. This means that two coupled nodes are linked through their 1st, $\ldots, q$th state variables. If there is an edge between node i and node j $(i \neq j)$, then $a_{ij} = a_{ji} = 1$; otherwise, $a_{ij} = a_{ji} = 0\,(i \neq j)$. If the degree k_i of node i is defined as the number of edge of node i, then

$$\sum_{\substack{j=1\\ j\neq i}}^{N} a_{ij} = \sum_{\substack{j=1\\ j\neq i}}^{N} a_{ji} = k_i, \qquad \text{and let} \qquad a_{ii} = -k_i, \quad i = 1, 2, \ldots, N.$$

Then equations (1.1) can be written as

$$\dot{\mathbf{x}}_i = f(\mathbf{x}_i) + c\sum_{j=1}^{N} a_{ij}\Gamma\mathbf{x}_j, \quad i = 1, 2, \ldots, N. \tag{1.2}$$

Suppose that the network is connected in the sense that there are no isolate clusters, then the coupling matrix $\mathbf{A} = (a_{ij}) \in \mathbb{R}^{N\times N}$ is a symmetric and irreducible matrix. It is well known that zero is an eigenvalue of $\mathbf{A}$ with multiplicity 1 and all the other eigenvalues of $\mathbf{A}$ are strictly negative (denoted by $0 = \lambda_1 > \lambda_2 \geqslant \lambda_3 \geqslant \cdots \geqslant \lambda_N$).

In Wang and Chen's paper [17], the dynamical system (1.2) is said to achieve (asymptotical) synchronization if

$$\mathbf{x}_i(t) \to \mathbf{s}(t), \qquad t \to \infty, \qquad i = 1, 2, \ldots, N, \tag{1.3}$$

where $\mathbf{s}(t) \in \mathbb{R}^d$ is a solution of an isolate node, namely,

$$\dot{\mathbf{s}}(t) = f(\mathbf{s}(t)). \tag{1.4}$$

The main mathematical results are given in [10, 17].

Theorem 1.1. *(Lemma 2 in [17] and theorem 1 in [10].) Suppose that there exists a $d \times d$ diagonal matrix $\Lambda > 0$, and two constants $\bar{k} < 0$ and $\tau > 0$, such that*

$$[Df(\mathbf{s}(t)) + k\Gamma]^T\Lambda + \Lambda[Df(\mathbf{s}(t)) + k\Gamma] \leqslant -\tau I_d \tag{1.5}$$

for all $k \leqslant \bar{k}$, where $I_d \in \mathbb{R}^{d\times d}$ is a unit matrix. If

$$c \geqslant \left|\frac{\bar{k}}{\lambda_2}\right|, \tag{1.6}$$

then the synchronization states (1.3) are exponentially stable.

Moreover, if each individual d-dimensional dynamical system is chaotic, let the maximum Lyapunov exponent of the individual node be $h_{\max} > 0$, then

$$c \geqslant \frac{h_{\max}}{|\lambda_2|}, \tag{1.7}$$

which implies that the synchronization states (1.3) are exponentially stable.

However, there always exist all kinds of random noises in the real world. So, it is natural to question that *does the deterministic dynamical system on network (1.1) effectively and essentially model the synchronization phenomenon in the real world, in other words, how does the random perturbation influence on the behavior of (1.1)?*

In this paper, we focus on investigating some mathematical results of synchronization of (1.1) under the influence of random noises, which are introduced in the following Ito's stochastic differential equation (SDE):

$$\mathrm{d}X_t^{i,\epsilon} = f\left(X_t^{i,\epsilon}\right)\mathrm{d}t + c\sum_{j=1}^{N} a_{ij}\Gamma X_t^{j,\epsilon}\,\mathrm{d}t + \sqrt{\epsilon_1}\,\mathrm{d}W_t + \sqrt{\epsilon_2}\sigma\left(X_t^{i,\epsilon}\right)\mathrm{d}B_t^i, \qquad i = 1, 2, \ldots, N, \tag{1.8}$$

where c, Γ and (a_{ij}) are defined as the same as those in (1.1), $(W_t)_{t\geqslant 0}$ is a d-dimensional Brownian motion, $\left(B_t^1\right)_{t\geqslant 0}, \left(B_t^2\right)_{t\geqslant 0}, \ldots, \left(B_t^N\right)_{t\geqslant 0}$ are r-dimensional Brownian motions, and all of them are mutually independent. $(W_t)_{t\geqslant 0}$ in (1.8) describes the random noise of exterior environment imposing on the whole dynamical network, and $\left(B_t^i\right)_{t\geqslant 0}$ represents the interior random fluctuation of the dynamical system in the node i. ϵ_1, ϵ_2 are the intensity of the random noises $(W_t)_{t\geqslant 0}$ and $\left(B_t^i\right)_{t\geqslant 0}$, respectively. However, stochastic processes with internal noise cannot always be written in the above Langevin form. This paper is restricted in considering form (1.8), because some tools of stochastic analysis can be conveniently used here (see [8, 21]).

Recently, there are some works studying the synchronization on the coupled network with random factors by the approach of stochastic analysis, see [4, 5, 15, 16]. For example, in [4], Berglund, Fernandez and Gentz consider SDE (1.8) of a one-dimension N periodic chain coupled with its nearest neighbors, of which $\epsilon_1 = 0$ and $f(x)$ is the negative gradient of a bistable potential $\frac{1}{4}x^4 - \frac{1}{2}x^2$. They obtain the precise (asymptotic) synchronization–desynchronization threshold estimates by some sophisticated mathematical theories, such as metastability and twist map of the integrable system. In [15, 16], Qian and Wang, Qian and Zhang study the frequency locking (frequency synchronization) of coupled oscillators under white noises in the view of non-equilibrium physics, such as entropy production. Deng, Ding and Feng consider the case of the stochastic coupled matrix, and some general approach is presented in [5].

For discussing synchronization of (1.2), two aspects are indicated in theorem 1.1, one is that the long time behavior of all nodes converges to an identical orbit, the so-called being at the synchronized state and this convergence is determined by the characteristics of individual node's dynamics and the topology of the entire coupled network; the other is that the identical orbit is the same as $\mathbf{s}(t)$, the solution of the equation of an isolate node (1.4). Therefore, in the analogical spirit of theorem 1.1, the main purposes of this paper are to investigate the following questions:

(i) Is there a synchronized threshold similar to (1.6) or (1.7) in theorem 1.1, which is determined by the characteristics of individual node's stochastic dynamics and the topology of the entire coupled network? How to understand the synchronization with random noise?

(ii) What is the identical orbit at the synchronized state in equation (1.8)? Is there some relationship between this identical orbit and asymptotical behavior of the SDE on individual node

$$\mathrm{d}X_t^{\epsilon} = f\left(X_t^{\epsilon}\right) + \sqrt{\epsilon_1}\,\mathrm{d}W_t + \sqrt{\epsilon_2}\sigma\left(X_t^{\epsilon}\right)\mathrm{d}B_t? \tag{1.9}$$

(iii) As $\epsilon_1, \epsilon_2 \to 0$, does the synchronized state in equation (1.8) converge to that of the corresponding dynamical network without random perturbation, as described in equation (1.2)?

Through some techniques of stochastic analysis, loosely speaking, the main results in this paper are: (1) for the general $f(x)$ and connectivity of the network in (1.8), some qualitative conditions of synchronization are given; (2) for some special classes of (1.8), even for a concrete $f(x)$ and topology of the network, some properties are shown in detail to answer (ii) and (iii) positively or negatively.

This paper is organized as follows. The main results are given in section 2, which contains three subsections. In section 2.1, similar to Lyapunov's direct method (see [20]) or the global stability analysis (see [2]), a synchronization threshold is given by employing Ito's formula. In section 2.2, question (ii) is discussed in different cases: (1) $\epsilon_1 > 0$ and $\epsilon_2 = 0$; (2) $\epsilon_1 = 0$ and $\epsilon_2 > 0$. These results demonstrate that the exterior environment noise of the whole systems and interior noise in each node bring the distinct effect on the synchronized states. In section 2.3, in order to compare the synchronized behavior of the coupled dynamical network with small random perturbation ($\epsilon_1, \epsilon_2 \to 0$) and the deterministic coupled dynamical network, a class of the reversible SDE with invariant probability measure (corresponding to the gradient system in deterministic dynamics, which is a very special class of dynamical systems) is considered particularly here. This simple and basic model gives some flavor of how the small random noises give rise to the synchronization (desynchronization) on the desynchronized (synchronized) coupled dynamical network. Finally, the proofs of lemmas and theorems in section 2 and some useful lemmas are shown in section 3.

2. Main results

Let

$$M = \mathbf{A} \otimes \Gamma = \begin{pmatrix} a_{11}\Gamma & a_{12}\Gamma & \ldots & a_{1N}\Gamma \\ a_{21}\Gamma & a_{22}\Gamma & \ldots & a_{2N}\Gamma \\ \ldots & \ldots & \ldots & \ldots \\ a_{N1}\Gamma & a_{N2}\Gamma & \ldots & a_{NN}\Gamma \end{pmatrix},$$

then equations (1.8) can be written as

$$\mathrm{d}\mathbf{X}_t^\epsilon = \mathbf{f}(\mathbf{X}_t^\epsilon)\,\mathrm{d}t + cM\mathbf{X}_t^\epsilon \mathrm{d}t + \sqrt{\epsilon_1}\,\mathrm{d}\mathbf{W}_t + \sqrt{\epsilon_2}\sigma(\mathbf{X}_t^\epsilon)\,\mathrm{d}\mathbf{B}_t, \tag{2.1}$$

where $\mathbf{X}_t^\epsilon = (X_1^\epsilon, \ldots, X_N^\epsilon)^T$, $\mathbf{f}(\mathbf{x}) = (f(x_1), f(x_2), \ldots, f(x_N))^T$, $\mathbf{W}_t = (W_t, \ldots, W_t)^T$, $\sigma(\mathbf{x}) = (\sigma(x_1), \sigma(x_2), \ldots, \sigma(x_N))^T$ and $\mathbf{B}_t = (B_t^1, \ldots, B_t^N)^T$. Denote $S = \{(x, x, \ldots, x) \in (\mathbb{R}^d)^N, x \in \mathbb{R}^d\}$, the diagonal of space $(\mathbb{R}^d)^N$, and S_δ is the δ neighborhood of S.

2.1. A synchronized threshold for random cases

In this subsection, some assumptions of f, Γ and σ are considered as follows:

(**A 2.1.1**) f is a global Lipschitz continuous function with Lipschitz constant K, i.e.

$$\|f(x) - f(y)\| \leqslant K\|x - y\|;$$

(**A 2.1.2**) $\Gamma = \mathrm{diag}(1, 1, \ldots, 1)$;

(**A 2.1.3**) σ is a bounded and global Lipschitz continuous $d \times r$-matrix-valued function, i.e.

$$|\sigma_{mn}(x) - \sigma_{mn}(y)| \leqslant l\|x - y\|, \qquad |\sigma_{mn}(x)| \leqslant L, \qquad 1 \leqslant m \leqslant d, \quad 1 \leqslant n \leqslant r.$$

Theorem 2.1. *If $c > \frac{K}{|\lambda_2|}$, then, for any $\delta > 0$, $r > 0$, and $\epsilon_1 > 0$,*

$$\lim_{\epsilon_2 \to 0} \lim_{t \to \infty} \sup_{\|\mathbf{x}\| \leqslant r} P\left(\mathbf{X}_t^\epsilon \in S_\delta \,\middle|\, \mathbf{X}_0^\epsilon = \mathbf{x}, \mathbf{x} \in (\mathbb{R}^d)^N\right) = 1. \tag{2.2}$$

Comparing with theorem 1.1, the synchronization in this theorem says that if the coupling strength c is larger than a threshold $\frac{K}{|\lambda_2|}$, then the system is synchronized whenever all the nodes remain closed to each other with high probability as ϵ_2 small enough. Moreover, if $\epsilon_2 = 0$, i.e. there is only exterior environment noise on the system. Thus, we have the following theorem:

Theorem 2.2. *If $c > \frac{K}{|\lambda_2|}$ and $\epsilon_2 = 0$, then for any $\epsilon_1 > 0$,*

$$X_t^{1,\epsilon_1} = X_t^{2,\epsilon_1} = \cdots = X_t^{N,\epsilon_1}, \qquad t \to \infty. \quad a.s.P$$

i.e. in probability 1, for any $\delta > 0$, there is a $t_0(\omega) > 0$, for any $t > t_0(\omega)$,

$$\mathbf{X}_t^{\epsilon} \in S_\delta.$$

In fact, this theorem can be regarded as a conclusion of theorem 3 in [20] and theorem 2.1 in [19], and shows that exterior environment noise on the global system does not destroy the synchronization.

A synchronized threshold $\frac{K}{|\lambda_2|}$ in the above theorems is determined by the global Lipschitz constant of the system on the individual node f and the eigenvalue λ_2 of the coupled matrix of the network. It seems that this threshold is rougher than those in equation (1.6), (1.7), since [10, 17] only analyze the local linear stability at $\mathbf{s}(t)$ (see equation (1.4)). Corresponding to theorem 1.1, Lyapunov exponent and the stability at a stationary solution in random dynamical systems can be found in [1, 13]. Some finer synchronized thresholds will be discussed in the authors' further research by utilizing the method of random dynamical systems.

2.2. Identical orbit at synchronized state

A subsequent important problem is to describe the identical orbit when the dynamical systems of all the nodes achieve synchronization. Since the long time behavior of the system with random noise is understood by its invariant measure, in this subsection, some relations of the identical orbit at synchronized state and the invariant probability measure of an individual uncoupled node (assuming this measure exists and is unique) are explored.

First, the effect of random noise of exterior environment $(W_t, t \geqslant 0)$ on the whole systems is considered, i.e. $\epsilon_2 = 0$ in (1.8),

$$\mathrm{d}X_t^{i,\epsilon_1} = f\left(X_t^{i,\epsilon_1}\right)\mathrm{d}t + c\sum_{j=1}^{N} a_{ij}\Gamma X_t^{j,\epsilon_1}\,\mathrm{d}t + \sqrt{\epsilon_1}\,\mathrm{d}W_t, \qquad i = 1, 2, \ldots, N, \tag{2.3}$$

and its corresponding equation on the individual uncoupled node is

$$\mathrm{d}\tilde{X}_t^{\epsilon_1} = f\left(\tilde{X}_t^{\epsilon_1}\right)\mathrm{d}t + \sqrt{\epsilon_1}\,\mathrm{d}W_t. \tag{2.4}$$

Besides f and Γ satisfying **A 2.1.1** and **A 2.1.2**, one assumes that equation (2.4) has a unique invariant probability measure $\tilde{\mu}_{\varepsilon_1}$. By the classic ergodic theory of diffusion processes (see [21]), this implies that for all $B \in \mathscr{B}(\mathbb{R}^d)$, Borel's σ fields on $\mathbb{R}^d$,

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \mathbf{1}_B\left(\tilde{X}_s^{\epsilon_1}\right)\mathrm{d}s = \tilde{\mu}_{\epsilon_1}(B). \tag{2.5}$$

Thus, one has

Theorem 2.3. *For any initial value of the solution of (2.3), in probability 1,*

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \mathbf{1}_B\left(X_s^{i,\epsilon_1}\right)\mathrm{d}s = \tilde{\mu}_{\epsilon_1}(B). \tag{2.6}$$

Although it is difficult to compare precisely the identical orbit in equation (2.3) and the orbit in equation (2.4) for any t large enough, the above theorem shows that the 'frequency' of the identical orbit visiting B converges to the invariant measure $\tilde{\mu}_{\epsilon_1}(B)$ as $t \to \infty$.

Second, for the coupled SDE

$$dX_t^{i,\epsilon_2} = f\left(X_t^{i,\epsilon_2}\right) dt + c\sum_{j=1}^{N} a_{ij}\Gamma X_t^{j,\epsilon_2}\, dt + \sqrt{\epsilon_2}\sigma\left(X_t^{i,\epsilon_2}\right) dB_t^i, \qquad i = 1, 2, \ldots, N, \tag{2.7}$$

and its corresponding equation on the individual uncoupled node

$$d\hat{X}_t^{\epsilon_2} = f\left(\hat{X}_t^{\epsilon_2}\right) dt + \sqrt{\epsilon_2}\sigma\left(\hat{X}_t^{\epsilon_2}\right) dB_t, \tag{2.8}$$

(i.e. $\epsilon_1 = 0, \epsilon_2 > 0$ in (1.8)), a natural problem is:

If (2.7) achieves synchronization, i.e. satisfies (2.2) in theorem 2, does

$$\lim_{\epsilon_2\to 0}\lim_{t\to\infty}\left[\frac{1}{t}\int_0^t \mathbf{1}_B\left(X_s^{i,\epsilon_2}\right) ds - \hat{\mu}_{\epsilon_2}(B)\right] = 0, \qquad i = 1, 2, \ldots, N \tag{2.9}$$

hold? where $\hat{\mu}_{\epsilon_2}$ is the unique invariant probability measure of the SDE (2.8).

The effect of the interior random fluctuation is much more complicated than that of the exterior random noise discussed above. In general, SDE (2.7) does not satisfy (2.9), for example:

Example 2.4. If $N = 2, d = 1, \sigma(x) = 1$, and $f(x) = -U'(x)$ in SDE (2.7), (2.8), where $U(x)$ satisfies $\int_{\mathbb{R}} \exp\left\{\frac{-2U(x)}{\epsilon_2}\right\} dx < \infty$, for any ϵ_2 small enough, then, it is easy to know that

$$\mu_{\epsilon_2}(dx_1\, dx_2) \equiv \bar{C}\exp\left\{-\frac{2U(x_1) + 2U(x_2) + c(x_1 - x_2)^2}{\epsilon_2}\right\} dx_1\, dx_2, \tag{2.10}$$

$$\hat{\mu}_{\epsilon_2}(dx) \equiv \hat{C}\exp\left\{-\frac{2U(x)}{\epsilon_2}\right\} dx \tag{2.11}$$

are the unique invariant measures of SDE (2.7), (2.8) respectively ($\bar{C}$ and $\hat{C}$ are the normalized constants).

Moreover, if one assumes that $U(x) \in C^3(\mathbb{R})$, and for $z_i \in \{z_1, z_2\} = \{z \in \mathbb{R}; U(z) = \min_{x\in\mathbb{R}} U(x)\}, U''(z_i) > 0, i = 1, 2$, then for any open neighborhood E_1 of z_1, satisfying $z_2 \notin E_1$, equality (7.5.22) at p436 in [21], and lemma 3.6 in section 3 imply

$$\lim_{\epsilon_2\to 0}\lim_{t\to\infty}\left[\frac{1}{t}\int_0^t \mathbf{1}_{E_1}\left(X_s^{i,\epsilon_2}\right) ds)\right] = \frac{[(U''(z_1))^2 + 2cU''(z_1)]^{-\frac{1}{2}}}{\sum_{i=1}^{2}[(U''(z_i))^2 + 2cU''(z_i)]^{-\frac{1}{2}}}, \tag{2.12}$$

and

$$\lim_{\epsilon_2\to 0}\hat{\mu}_{\epsilon_2}(E_1) = \frac{|U''(z_1)|^{-\frac{1}{2}}}{|U''(z_1)|^{-\frac{1}{2}} + |U''(z_2)|^{-\frac{1}{2}}}. \tag{2.13}$$

This shows that (2.9) does NOT hold.

However, if $z_1 = z_2$, obviously, (2.9) holds.

2.3. Comparison of synchronization between a coupled gradient system and its small random perturbation

It is a challenging problem in probability theory to compare the long time behavior of the dynamical system and its small random perturbation (see [7]). So, it is difficult to give a clear relation of synchronization between a coupled dynamical system and its small random perturbation. However, because the gradient dynamical system $\dot{x} = -\nabla U(x)$ is one of the simplest dynamical systems and its small random perturbation $\mathrm{d}x_t = -\nabla U(x_t)\,\mathrm{d}t + \sqrt{\epsilon_2}\,\mathrm{d}B_t$ has a unique invariant measure $\hat{C}\exp\left\{-\frac{2U(x)}{\epsilon_2}\right\}\mathrm{d}x$, if $\int\exp\left\{-\frac{2U(x)}{\epsilon_2}\right\}\mathrm{d}x < \infty$, in this subsection, a complete comparison of synchronization of the coupled gradient system

$$\dot{\mathbf{x}}_i = -\nabla U(\mathbf{x}_i) + c\sum_{j=1}^{N} a_{ij}\Gamma\mathbf{x}_j, \qquad i = 1, 2, \ldots, N, \tag{2.14}$$

and its small random perturbation

$$\mathrm{d}X_t^{i,\epsilon_2} = -\nabla U\left(X_t^{i,\epsilon_2}\right)\mathrm{d}t + c\sum_{j=1}^{N} a_{ij}\Gamma X_t^{j,\epsilon_2}\,\mathrm{d}t + \sqrt{\epsilon_2}I_d\,\mathrm{d}B_t^i, \qquad i = 1, 2, \ldots, N \tag{2.15}$$

are presented.

In this subsection, some assumptions of U, Γ are considered as follows:

(**A 2.3.1**) $U(x) \in C^2(\mathbb{R}^d, \mathbb{R}^1)$, $\int\exp\left\{-\frac{2U(x)}{\epsilon_2}\right\}\mathrm{d}x < \infty$, for any $\epsilon_2 > 0$, $\lim_{\|x\|\to\infty} U(x) = \infty$ and for any $r > 0$, $\{x \in \mathbb{R}^d \,|\, \|x\| \leqslant r\}$ is compact. For example, if for some $r > 0$, there is a constant $C > 0$, such that, for all $x \in \mathbb{R}^d \backslash B_{\mathbb{R}^d}(0, r)$, Hess $U(x) > CI_d$, then it is easy to show that the above assumptions of $U(x)$ hold, where Hess $U(x) \equiv \left(\frac{\partial^2 U(x)}{\partial x_i \partial x_j}\right)_{1\leqslant i,j\leqslant d}$, the Hessian matrix of U;

(**A 2.3.2**) The number of the elements in $\Xi \equiv \{x \in \mathbb{R}^d \mid U(x) = \min_{y\in\mathbb{R}^d} U(y)\}$, which is denoted by m, should be finite. Without any loss of generality, one assumes $\min_{y\in\mathbb{R}^d} U(y) = 0$;

(**A 2.3.3**) $\Gamma = \mathrm{diag}(\overbrace{\underbrace{1, 1, \ldots, 1}_{q}, 0, 0, \ldots, 0}^{d})$, $q \leqslant d$.

Let $H(\mathbf{x}) \equiv \sum_{i=1}^{N} U(\mathbf{x}_i) - \frac{c\mathbf{x}^T M\mathbf{x}}{2}$, $\mathbf{x} \in (\mathbb{R}^d)^N$, then, the number of the elements in

$$\Lambda \equiv \{\mathbf{x} \in (\mathbb{R}^d)^N | H(\mathbf{x}) = \min_{\mathbf{y}\in(\mathbb{R}^d)^N} H(\mathbf{y})\}$$

is also finite by assumption **A 2.3.2**, and it is easy to show that

$$\mu_{\epsilon_2}(\mathrm{d}\mathbf{x}) \equiv \tilde{C}\exp\left(-\frac{2H(\mathbf{x})}{\epsilon_2}\right)\mathrm{d}\mathbf{x}$$

is the unique invariant probability measure of (2.15).

Theorem 2.5. *For all $\delta > 0$, let Λ_δ denote the δ-neighborhood of Λ, then for any $r > 0$*

$$\lim_{\epsilon_2\to 0}\lim_{t\to\infty}\sup_{\|\mathbf{x}\|\leqslant r} P\left(\mathbf{X}_t^{\epsilon_2} \in \Lambda_\delta \middle| \mathbf{X}_0^{\epsilon_2} = \mathbf{x}, \mathbf{x} \in (\mathbb{R}^d)^N\right) = 1. \tag{2.16}$$

This theorem implies that the synchronization of the dynamical system (2.15) is determined by the location of Λ. Through analyzing the relation of m, Γ and the location of Ξ, the precise synchronized condition is obtained as follows:

Theorem 2.6. *The dynamical system (2.15) can achieve synchronization in the sense of (2.2), if one of the following three conditions is satisfied:*

(1) $m = 1$;

(2) $m > 1$ and $q = d$;

(3) $m > 1, q < d$, and for $z_k = (z_{k1}, z_{k2}, \ldots, z_{kd}) \in \Xi$, denoting $z_k^{(q)} = (z_{k1}, z_{k2}, \ldots, z_{kq})$, for any $k \neq l, k, l = 1, 2, \ldots, m$, $z_k^{(q)} \neq z_l^{(q)}$.

On the other hand, let $\tilde{\Lambda} \equiv \{\mathbf{z} \in \Xi^N \backslash S | z_i^{(q)} = z_j^{(q)}, \text{ if } a_{ij} \neq 0, i, j = 1, \ldots, N\}$. It is clear that $\tilde{\Lambda} \subset \Lambda$, $\tilde{\Lambda} \not\subset S$, if $m > 1$ and $q < d$. Moreover, assuming that μ_{ϵ_2} converges weakly to a probability measure μ, then, by [9], $\mu(\Lambda) = 1$. So,

Theorem 2.7. *If* $m > 1, q < d$ *and* $\mu(\tilde{\Lambda}) > 0$*, then there is a* $\delta_0 > 0$ *such that*

$$\liminf_{\epsilon_2 \to 0} \lim_{t \to \infty} \sup_{\|\mathbf{x}\| \leqslant r} P\left(\mathbf{X}_t^{\epsilon_2} \notin S_{\delta_0} \middle| \mathbf{X}_0^{\epsilon_2} = \mathbf{x}, \mathbf{x} \in (\mathbb{R}^d)^N\right) > 0. \tag{2.17}$$

Remark 2.3.1. Under **A 2.3.1**, similar to the section 2 in [9], μ_{ϵ_2} is tight and the limit measures μ satisfy $\mu(\Lambda) = 1$, where μ are not unique possibly. However, in some cases, it can be shown that μ is unique, for example: if $U(\mathbf{x}) \in C^3(\mathbb{R}^d, \mathbb{R}^1)$, and for any $\mathbf{x} \in \Lambda$, $\det \text{Hess}\, H(\mathbf{x}) \neq 0$, then μ is unique and

$$\mu(\mathbf{x}) = \frac{(\det \text{Hess}\, H(\mathbf{x}))^{-1/2}}{\sum_{\mathbf{y} \in \Lambda} (\det \text{Hess}\, H(\mathbf{y}))^{-1/2}},$$

where det represents the determinant of a matrix.

There are some simple but interesting conclusions obtained from theorem 2.6 and theorem 2.7.

(1) The synchronization of the stochastic gradient system (2.15) can be achieved for any coupled strength c in theorem 2.6, however, there is a synchronized threshold for a general f in theorem 2.1 in subsection 2.1.
(2) In [4], Berglund, Fernandez and Gentz show that the deterministic coupled gradient system cannot synchronize if the coupled strength is less than some critical value and this dynamical system starts at the neighborhood of a stable equilibrium point of $H(\mathbf{x})$ which is not in S. However, theorem 2.6 implies that if the stochastic coupled gradient system satisfies the conditions in the theorem, it always achieves synchronization (in the sense of (2.2)) for any coupled strength and any starting point. In fact, the model discussed in [4] satisfies the condition (2) of theorem 2.6.
(3) Theorem 2.7 demonstrates that in some special cases, the random noise can destroy the synchronization of (2.14).

3. Proofs

Let $\langle \cdot, \cdot \rangle_n$ denote the inner product and $\|\cdot\|_n$ denote the norm in $\mathbb{R}^n$, $n < \infty$.

3.1. Proofs of theorems 2.1 and 2.2

Before the proofs of theorems 2.1 and 2.2, some lemmas are given under (**A 2.1.2**)–(**A 2.1.3**) first.

Lemma 3.1. *For* $\mathbf{x} = (x_1, \ldots, x_N) \in (\mathbb{R}^d)^N$, $x_i \in \mathbb{R}^d$, $i = 1, \ldots, N$,

$$\langle \mathbf{x}, -M\mathbf{x} \rangle_{dN} = \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N a_{ij} \|\mathbf{x}_i - \mathbf{x}_j\|_d^2, \tag{3.1}$$

$$\langle \mathbf{x}, -M\mathbf{f}(\mathbf{x}) \rangle_{dN} \leqslant K \langle \mathbf{x}, -M\mathbf{x} \rangle_{dN}, \tag{3.2}$$

$$|\lambda_2| \langle \mathbf{x}, -M\mathbf{x} \rangle_{dN} \leqslant \langle M\mathbf{x}, M\mathbf{x} \rangle_{dN} \leqslant |\lambda_N| \langle \mathbf{x}, -M\mathbf{x} \rangle_{dN}. \tag{3.3}$$

Proof. Noting $\sum_{i=1}^{N} a_{ij} = 0$ and $a_{ij} = a_{ji}$, for $\mathbf{x}, \mathbf{y} \in (\mathbb{R}^d)^N$,

$$\begin{aligned}\langle \mathbf{x}, M\mathbf{y}\rangle_{dN} &= \langle \mathbf{x}, (\mathbf{A}\otimes\Gamma)\mathbf{y}\rangle_{dN}\\ &= \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\langle x_i, (y_j - y_i)\rangle_d = -\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\langle x_j, (y_j - y_i)\rangle_d\\ &= -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\langle (x_j - x_i), (y_j - y_i)\rangle_d. \end{aligned}\tag{3.4}$$

It is clear that (3.1) holds, and by the Cauchy–Schwarz inequality,

$$\langle \mathbf{x}, -M\mathbf{f}(\mathbf{x})\rangle_{dN} = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\langle (x_i - x_j), (f(x_i) - f(x_j))\rangle_d \leqslant K\langle \mathbf{x}, -M\mathbf{x}\rangle_{dN}.$$

Since $0 > \lambda_2 \geqslant \cdots \geqslant \lambda_N$ are the eigenvalues of $\mathbf{A}$, so do M. Therefore,

$$|\lambda_2|\langle \mathbf{x}, -M\mathbf{x}\rangle_{dN} \leqslant \langle M\mathbf{x}, M\mathbf{x}\rangle_{dN} = \langle \mathbf{x}, M^2\mathbf{x}\rangle_{dN} \leqslant |\lambda_N|\langle \mathbf{x}, -M\mathbf{x}\rangle_{dN}.$$

□

Lemma 3.2. *Let $\left(\mathbf{X}_t^\epsilon, t \geqslant 0\right)$ be the solution of SDE (2.1) with the initial value $\mathbf{x} = (x_1, \ldots, x_N) \in \left(\mathbb{R}^d\right)^N$, then $\int_0^t \left\langle M\mathbf{X}_s^\epsilon, \sigma\left(\mathbf{X}_s^\epsilon\right) \mathrm{d}\mathbf{B}_s\right\rangle_{dN}$ is a $(\mathcal{F}_t)$ martingale, where $\mathcal{F}_t \equiv \sigma(\mathbf{B}_s, \mathbf{W}_s, s \leqslant t)$. This means that $E\left(\int_0^t \left\langle M\mathbf{X}_s^\epsilon, \sigma\left(\mathbf{X}_s^\epsilon\right) \mathrm{d}\mathbf{B}_s\right\rangle_{dN}\right) = 0$.*

Proof. Because all the coefficients of SDE (2.1) are global Lipschitz continuous and the initial value is deterministic, the classic proof of existence and uniqueness of the SDE imply that the lemma holds (see [8]). □

Proof of theorem 2.1. By Ito's formula,

$$\begin{aligned} d\left\langle \mathbf{X}_t^\epsilon, -M\mathbf{X}_t^\epsilon\right\rangle_{dN} &= 2\left\langle \mathbf{X}_t^\epsilon, -M\,\mathrm{d}\mathbf{X}_t^\epsilon\right\rangle_{dN} + \epsilon_2\,\mathrm{tr}\left(-M\sigma\left(\mathbf{X}_t^\epsilon\right)\sigma\left(\mathbf{X}_t^\epsilon\right)^T\right)\mathrm{d}t\\ &= 2\left\langle \mathbf{X}_t^\epsilon, -M\mathbf{f}\left(\mathbf{X}_t^\epsilon\right)\right\rangle_{dN}\mathrm{d}t - 2\left\langle M\mathbf{X}_t^\epsilon, cM\mathbf{X}_t^\epsilon\right\rangle_{dN}\mathrm{d}t - 2\left\langle \mathbf{X}_t^\epsilon, \sqrt{\epsilon_1}M\,\mathrm{d}\mathbf{W}_t\right\rangle_{dN}\\ &\quad - 2\left\langle \mathbf{X}_t^\epsilon, \sqrt{\epsilon_2}M\sigma\left(\mathbf{X}_t^\epsilon\right)\mathrm{d}\mathbf{B}_t\right\rangle_{dN} + \epsilon_2\,\mathrm{tr}\left(-M\sigma\left(\mathbf{X}_t^\epsilon\right)\sigma\left(\mathbf{X}_t^\epsilon\right)^T\right)\mathrm{d}t, \end{aligned}\tag{3.5}$$

(3.4) implies, $\left\langle \mathbf{X}_t^\epsilon, M\,\mathrm{d}\mathbf{W}_t\right\rangle_{dN} = -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\left\langle \mathbf{X}_t^{i,\epsilon} - \mathbf{X}_t^{j,\epsilon}, (\mathrm{d}W_t - \mathrm{d}W_t)\right\rangle_{dN} = 0$. Because $|\sigma_{mn}| < L$, lemmas 3.1 and 3.2 imply,

$$E^{\mathbf{x}}\left\langle \mathbf{X}_t^\epsilon, -M\mathbf{X}_t^\epsilon\right\rangle_{dN} \leqslant \langle \mathbf{x}, -M\mathbf{x}\rangle_{dN} + \int_0^t (2K - 2c|\lambda_2|)E^{\mathbf{x}}\left\langle \mathbf{X}_s^\epsilon, -M\mathbf{X}_s^\epsilon\right\rangle_{dN}\mathrm{d}s + Nd^2L^2\epsilon_2\,\mathrm{tr}(-M)t,$$

hence, by the comparison principle

$$\begin{aligned} E^{\mathbf{x}}\left\langle \mathbf{X}_t^\epsilon, -M\mathbf{X}_t^\epsilon\right\rangle_{dN} &\leqslant \left[\langle \mathbf{x}, -M\mathbf{x}\rangle_{dN} + \frac{Nd^2L^2\epsilon_2\,\mathrm{tr}(-M)}{(2K - 2c|\lambda_2|)}\right]\\ &\quad \times \exp[(2K - 2c|\lambda_2|)t] - \frac{Nd^2L^2\epsilon_2\,\mathrm{tr}(-M)}{(2K - 2c|\lambda_2|)}. \end{aligned}$$

Since $c > \frac{K}{|\lambda_2|}$, it follows that for any $r > 0$

$$\begin{aligned} \lim_{\epsilon_2\to 0}\lim_{t\to\infty}\sup_{\|\mathbf{x}\|\leqslant r}\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}E\left\|\mathbf{X}_t^{i,\epsilon} - \mathbf{X}_t^{j,\epsilon}\right\|_d^2 &= \lim_{\epsilon_2\to 0}\lim_{t\to\infty}\sup_{\|\mathbf{x}\|\leqslant r} E^{\mathbf{x}}\left\langle \mathbf{X}_t^\epsilon, -M\mathbf{X}_t^\epsilon\right\rangle_{dN}\\ &= \lim_{\epsilon_2\to 0}\frac{Nd^2L^2\epsilon_2\,\mathrm{tr}(-M)}{-(2K - 2c|\lambda_2|)} = 0. \end{aligned}$$

So, the theorem holds by Chebyshev's inequality. □

Proof of theorem 2.2. If $\epsilon_2 = 0$, the model (2.1) can be written in the following form:

$$\mathrm{d}\mathbf{X}_t^{\epsilon_1} = f\left(\mathbf{X}_t^{\epsilon_1}\right)\mathrm{d}t + cM\mathbf{X}_t^{\epsilon_1}\mathrm{d}t + \sqrt{\epsilon_1}\,\mathrm{d}\mathbf{W}_t. \tag{3.6}$$

It is easy to compute that in probability 1,

$$\begin{aligned} d\left\langle\mathbf{X}_t^{\epsilon_1}, -M\mathbf{X}_t^{\epsilon_1}\right\rangle_{dN} &= 2\left\langle\mathbf{X}_t^{\epsilon_1}, -Mf\left(\mathbf{X}_t^{\epsilon_1}\right)\right\rangle_{dN}\mathrm{d}t - 2\left\langle M\mathbf{X}_t^{\epsilon_1}, cM\mathbf{X}_t^{\epsilon_1}\right\rangle_{dN}\mathrm{d}t \\ &\leqslant (2K - 2c|\lambda_2|)\left\langle\mathbf{X}_t^{\epsilon_1}, -M\mathbf{X}_t^{\epsilon_1}\right\rangle_{dN}\mathrm{d}t. \end{aligned}$$

Therefore, under the condition $c > \frac{K}{|\lambda_2|}$, for any ϵ_1, as $t \to \infty$,

$$\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\left\|\mathbf{X}_t^{i,\epsilon} - \mathbf{X}_t^{j,\epsilon}\right\|_d^2 = \left\langle\mathbf{X}_t^{\epsilon_1}, -M\mathbf{X}_t^{\epsilon_1}\right\rangle_{dN} \leqslant \langle\mathbf{x}, -M\mathbf{x}\rangle_{dN}\exp[(2K - 2c|\lambda_2|)t] \to 0. \tag{3.7}$$

Theorem 2.2 follows. □

3.2. *Proof of theorem 2.3*

Let $\left(\mathbf{X}_t^{\epsilon_1} \equiv \left(X_t^{1,\epsilon_1}, \ldots, X_t^{N,\epsilon_1}\right)^T, t \geqslant 0\right)$ be the solution of (2.3) with the initial value $\mathbf{x} \in (\mathbb{R}^d)^N$.

Lemma 3.3. *For any* $i = 1, \ldots, N$,

$$E^{\mathbf{x}}\left[\exp\left(\frac{1}{2}\int_0^\infty\left\langle\frac{c}{\sqrt{\epsilon_1}}\sum_{i=1}^{N}a_{ij}\Gamma X_t^{j,\epsilon_1}, \frac{c}{\sqrt{\epsilon_1}}\sum_{i=1}^{N}a_{ij}\Gamma X_t^{j,\epsilon_1}\right\rangle_d\mathrm{d}t\right)\right] < \infty, \tag{3.8}$$

$$E^{\mathbf{x}}\left[\int_0^\infty\left\langle\frac{c}{\sqrt{\epsilon_1}}\sum_{i=1}^{N}a_{ij}\Gamma X_t^{j,\epsilon_1}, \frac{c}{\sqrt{\epsilon_1}}\sum_{i=1}^{N}a_{ij}\Gamma X_t^{j,\epsilon_1}\right\rangle_d\mathrm{d}t\right] < \infty. \tag{3.9}$$

Proof of lemma 3.3. By the right-hand side of inequality (3.3) in lemma 3.1 and (3.5) in the proof of lemma 3.2, noting $c > \frac{K}{|\lambda_2|}$, then

$$\begin{aligned} (3.8) &= E^{\mathbf{x}}\left[\exp\left(\frac{c^2}{2\epsilon_1}\int_0^\infty\left\langle\left(M\mathbf{X}_t^{\epsilon_1}\right)_i, \left(M\mathbf{X}_t^{\epsilon_1}\right)_i\right\rangle_d\mathrm{d}t\right)\right] \\ &\leqslant E^{\mathbf{x}}\left[\exp\left(\frac{c^2|\lambda_N|}{2\epsilon_1}\int_0^\infty\left\langle\mathbf{X}_t^{\epsilon_1}, -M\mathbf{X}_t^{\epsilon_1}\right\rangle_{dN}\mathrm{d}t\right)\right] \\ &\leqslant \exp\left(\frac{c^2|\lambda_N|}{2\epsilon_1}\langle\mathbf{x}, -M\mathbf{x}\rangle_{dN}\int_0^\infty\exp[(2K - 2c|\lambda_2|)t]\,\mathrm{d}t\right) \\ &< \infty. \end{aligned} \tag{3.10}$$

$$\begin{aligned} (3.9) &= E^{\mathbf{x}}\left[\left(\frac{c^2}{\epsilon_1}\int_0^\infty\left\langle\left(M\mathbf{X}_t^{\epsilon_1}\right)_i, \left(M\mathbf{X}_t^{\epsilon_1}\right)_i\right\rangle_d\mathrm{d}t\right]\right. \\ &\leqslant \frac{c^2|\lambda_N|}{\epsilon_1}\langle\mathbf{x}, -M\mathbf{x}\rangle_{dN}\int_0^\infty\exp[(2K - 2c|\lambda_2|)t]\,\mathrm{d}t \\ &< \infty. \end{aligned} \tag{3.11}$$

□

Lemma 3.4. *For* $i = 1, \ldots, N$, *let*

$$Z_t^{i,\epsilon_1} \equiv \exp\left(-\int_0^t \left\langle \frac{c}{\sqrt{\epsilon_1}} \sum_{i=1}^N a_{ij}\Gamma X_s^{j,\epsilon_1},\ \mathrm{d}W_s \right\rangle_d - \frac{1}{2}\int_0^t \left\langle \frac{c}{\sqrt{\epsilon_1}} \sum_{i=1}^N a_{ij}\Gamma X_s^{j,\epsilon_1},\ \frac{c}{\sqrt{\epsilon_1}} \sum_{i=1}^N a_{ij}\Gamma X_s^{j,\epsilon_1} \right\rangle_d \mathrm{d}s\right), \tag{3.12}$$

then, $\left(Z_t^{i,\epsilon_1}, \sigma(W_s, s \leqslant t)\right)$ *is a closed martingale on* $[0, \infty]$, *and* $0 < Z_\infty^{i,\epsilon_1} < \infty$ *in probability 1.*

Proof of lemma 3.4. Noting that

$$\begin{aligned}
&\sup_{t\geqslant 0} E^{\mathbf{x}} \left| \int_0^t \left\langle \frac{c}{\sqrt{\epsilon_1}} \sum_{i=1}^N a_{ij}\Gamma X_s^{j,\epsilon_1},\ \mathrm{d}W_s \right\rangle_d \right|^2 \\
&= \sup_{t\geqslant 0} E^{\mathbf{x}} \left(\int_0^t \left\langle \frac{c}{\sqrt{\epsilon_1}} \sum_{i=1}^N a_{ij}\Gamma X_s^{j,\epsilon_1},\ \frac{c}{\sqrt{\epsilon_1}} \sum_{i=1}^N a_{ij}\Gamma X_s^{j,\epsilon_1} \right\rangle_d \mathrm{d}s\right) \\
&\leqslant \frac{c^2|\lambda_N|}{\epsilon_1} \langle \mathbf{x}, -M\mathbf{x}\rangle_{dN} \int_0^\infty \exp[(2K - 2c|\lambda_2|)t]\,\mathrm{d}t \\
&< \infty,
\end{aligned} \tag{3.13}$$

it shows that $\left(\int_0^t \left\langle \frac{c}{\sqrt{\epsilon_1}} \sum_{i=1}^N a_{ij}\Gamma X_s^{j,\epsilon_1},\ \mathrm{d}W_s\right\rangle_d, \sigma(W_s, s \leqslant t), 0 \leqslant t \leqslant \infty\right)$ is a uniformly square integrable martingale on $[0, \infty]$ and

$$\begin{aligned}
&E^{\mathbf{x}} \left(\left| \int_0^\infty \left\langle \frac{c}{\sqrt{\epsilon_1}} \sum_{i=1}^N a_{ij}\Gamma X_s^{j,\epsilon_1},\ \mathrm{d}W_s \right\rangle_d \right|^2 \right) \\
&= E^{\mathbf{x}} \left(\int_0^\infty \left\langle \frac{c}{\sqrt{\epsilon_1}} \sum_{i=1}^N a_{ij}\Gamma X_s^{j,\epsilon_1},\ \frac{c}{\sqrt{\epsilon_1}} \sum_{i=1}^N a_{ij}\Gamma X_s^{j,\epsilon_1} \right\rangle_d \mathrm{d}s\right) \\
&< \infty.
\end{aligned} \tag{3.14}$$

So, lemma (3.3) and (3.14) imply $\left(Z_t^{i,\epsilon_1}, \sigma(W_s, s \leqslant t)\right)$ is a closed martingale on $[0, \infty]$ by Novikov's condition (see [8]).

Moreover, (3.14) means that

$$\left| \int_0^\infty \left\langle \frac{c}{\sqrt{\epsilon_1}} \sum_{i=1}^N a_{ij}\Gamma X_s^{j,\epsilon_1},\ \mathrm{d}W_s \right\rangle \right| < \infty,$$

$$\left| \int_0^\infty \left\langle \frac{c}{\sqrt{\epsilon_1}} \sum_{i=1}^N a_{ij}\Gamma X_s^{j,\epsilon_1},\ \frac{c}{\sqrt{\epsilon_1}} \sum_{i=1}^N a_{ij}\Gamma X_s^{j,\epsilon_1} \right\rangle_d \mathrm{d}s \right| < \infty$$

in probability 1. Therefore, $0 < Z_\infty^{i,\epsilon_1} < \infty$ in probability 1. □

Proof of theorem 2.3. Let $(W_t, t \geqslant 0)$ is a d-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$.

Denote $\tilde{P}^{\mathbf{x}}(\cdot) \equiv E^{\mathbf{x}}\left(\mathbf{1}.Z_\infty^{i,\epsilon_1}\right)$, a new probability measure on $(\Omega, \mathcal{F}_\infty)$, where $\cdot \in \mathcal{F}_\infty$, and $\mathcal{F}_\infty \equiv \sigma(W_t, t \leqslant \infty)$.

Consider SDE (2.3), (2.4)

$$\mathrm{d}X_t^{i,\epsilon_1} = f\left(X_t^{i,\epsilon_1}\right)\mathrm{d}t + c\sum_{j=1}^{N} a_{ij}\Gamma X_t^{j,\epsilon_1}\,\mathrm{d}t + \sqrt{\epsilon_1}\,\mathrm{d}W_t, \tag{3.15}$$

$$\mathrm{d}\tilde{X}_t^{\epsilon_1} = f\left(\tilde{X}_t^{\epsilon_1}\right)\mathrm{d}t + \sqrt{\epsilon_1}\,\mathrm{d}W_t, \tag{3.16}$$

lemma 3.4 means that $\left(X_t^{i,\epsilon_1}, t \geqslant 0\right)$ is the weak solution of SDE (2.4) (or (3.16)) under $\tilde{p}^{\mathbf{x}}(\cdot)$ by the Cameron–Martin–Girsanov theorem (see [8, 21]). For any $B \in \mathscr{B}(\mathbb{R}^d)$, let

$$\tilde{O} \equiv \left\{\omega \middle| \lim_{t\to\infty}\frac{1}{t}\int_0^t \mathbf{1}_B\left(\tilde{X}_s^{\epsilon_1}\right)\mathrm{d}s = \tilde{\mu}_{\epsilon_1}(B)\right\},$$

$$O \equiv \left\{\omega \middle| \lim_{t\to\infty}\frac{1}{t}\int_0^t \mathbf{1}_B\left(X_s^{i,\epsilon_1}\right)\mathrm{d}s = \tilde{\mu}_{\epsilon_1}(B)\right\},$$

it is clear that $\tilde{O} \in \mathcal{F}_\infty$, $O \in \mathcal{F}_\infty$ and $P^{\mathbf{x}}(\tilde{O}^c) = 0$ by (2.5). Thus

$$P^{\mathbf{x}}(\tilde{O}^c) = \tilde{P}^{\mathbf{x}}(O^c) = E^{\mathbf{x}}\left(\mathbf{1}_{O^c} Z_\infty^{i,\epsilon_1}\right) = 0. \tag{3.17}$$

Since lemma 3.4 claims that $0 < Z_\infty^{i,\epsilon_1} < \infty$ in probability 1, this implies that $P^{\mathbf{x}}(O^c) = 0$, i.e.

$$P^{\mathbf{x}}\left(\lim_{t\to\infty}\frac{1}{t}\int_0^t \mathbf{1}_B\left(X_s^{i,\epsilon_1}\right)\mathrm{d}s = \tilde{\mu}_{\epsilon_1}(B)\right) = 1. \tag{3.18}$$

□

3.3. Proofs of theorems 2.5, 2.6 and 2.7

Lemma 3.5. *For any* $\epsilon_2 > 0$, $\int_{(\mathbb{R}^d)^N}\exp\left\{-\frac{H(\mathbf{x})}{\epsilon_2}\right\}\mathrm{d}\mathbf{x} < \infty$, *and for any* $r > 0$, $\{\mathbf{x} \in (\mathbb{R}^d)^N | H(\mathbf{x}) \leqslant r\}$ *is compact.*

Proof. Since $H(\mathbf{x}) \geqslant \sum_{i=1}^N U(x_i)$, by **A 2.3.1**, the lemma follows. □

Lemma 3.6. *The family of probability measures* $(\mu_{\epsilon_2}, \epsilon_2 > 0)$ *is tight, and for any* $\delta > 0$,

$$\lim_{\epsilon_2\to 0}\mu_{\epsilon_2}(\mathbf{x} \in (\mathbb{R}^d)^N | H(\mathbf{x}) \leqslant \delta) = 1. \tag{3.19}$$

This means that if μ' *is a weak limiting measure of* $\left(\mu_{\epsilon_2}\right)$, *then* $\mu'(\Lambda) = 1$.

The proof of this lemma is similar to the content of section 2 in [9], if lemma 3.5 holds. So, it is omitted here.

Proof of theorem 2.5. By corollary 7.5.24 in [21],

$$\lim_{t\to\infty}\sup_{\|\mathbf{x}\|\leqslant r} P\left(\mathbf{X}_t^{\epsilon_2} \in \Lambda_\delta \middle| \mathbf{X}_0^{\epsilon_2} = \mathbf{x}, \mathbf{x} \in (\mathbb{R}^d)^N\right) = \mu_{\epsilon_2}(\Lambda_\delta), \tag{3.20}$$

and by lemma 3.6, $\lim_{\epsilon_2\to 0}\mu_{\epsilon_2}(\Lambda_\delta) = 1$, the theorem follows. □

Proof of theorem 2.6. Noting that

$$H(\mathbf{x}) = \sum_{i=1}^N U(x_i) + \frac{1}{2}\sum_{i=1}^N\sum_{j>i}^N a_{ij}\left[\sum_{k=1}^q \left(x_{i_k} - x_{j_k}\right)^2\right] \geqslant 0,$$

it is easy to check that if U satisfies one of (1), (2), (3) in theorem 2.6, then $\Lambda \in S$. So, theorem 2.5 implies the results. □

Proof of theorem 2.7. By lemma 3.6, for any open δ-neighborhood $\tilde{\Lambda}_\delta$ of $\tilde{\Lambda}$

$$\liminf_{\epsilon_2 \to 0} \mu_{\epsilon_2}(\tilde{\Lambda}_\delta) \geqslant \mu(\tilde{\Lambda}_\delta) \geqslant \mu(\tilde{\Lambda}) > 0.$$

If $m > 1, q < d$, then $\tilde{\Lambda} \cap S = \emptyset$. So, there is a $\delta_0 > 0$ such that $\tilde{\Lambda}_{\delta_0} \cap S_{\delta_0} = \emptyset$, thus

$$\begin{aligned}
&\liminf_{\epsilon_2 \to 0} \lim_{t\to\infty} \sup_{\|\mathbf{x}\|\leqslant r} P\left(\mathbf{X}_t^{\epsilon_2} \notin S_{\delta_0} \middle| \mathbf{X}_0^{\epsilon_2} = \mathbf{x}, \mathbf{x} \in (\mathbb{R}^d)^N\right) \\
&\geqslant \liminf_{\epsilon_2 \to 0} \lim_{t\to\infty} \sup_{\|\mathbf{x}\|\leqslant r} P\left(\mathbf{X}_t^{\epsilon_2} \in \tilde{\Lambda}_{\delta_0} \middle| \mathbf{X}_0^{\epsilon_2} = \mathbf{x}, \mathbf{x} \in (\mathbb{R}^d)^N\right) \\
&\geqslant \mu(\tilde{\Lambda}) \\
&> 0.
\end{aligned}$$

□

Acknowledgments

One of the authors, YL would like to acknowledge the financial supports of NSFC (No. 10531070), SRF for ROCS, and Science and Technology Ministry 973 project(2006CB805900). Both of the authors would like to thank Ms Juan Lei for useful discussion and who contributed to subsection 2.3 as part of her dissertation for her Master's degree. The authors also wish to thank the anonymous referee for giving useful suggestions.

References

[1] Arnold L 1998 *Random Dynamical Systems* (Berlin: Springer)
[2] Atay F M and Jost J 2004 Delays, connection topology, and synchronization of coupled chaotic maps *Phys. Rev. Lett.* **92** 144101
[3] Barahona M and Pecora L M 2002 Synchronization in small-world systems. *Phys. Rev. Lett.* **89** 054101
[4] Berglund N, Fernandez B and Gentz B 2007 Metastability in interacting nonlinear stochastic differential equations: I. From weak coupling to synchronization *Nonlinearity* **20** 2551–81
[5] Deng Y C, Ding M Z and Feng J F 2004 Synchronization in stochastic coupled systems: theoretical results *J. Phys. A: Math. Gen.* **37** 2163–73
[6] Fan J and Wang X F 2005 On synchronization in scale-free dynamical networks *Physica* A **349** 443–51
[7] Freidlin M I and Wentzell A D 1984 *Random Perturbations of Dynamical Systems* (Berlin: Springer)
[8] Ikeda N and Watanabe S 1989 *Stochastic Differential Equations and Diffusion Processes* (North-Holland: Amsterdam)
[9] Hwang C R 1980 Laplaces method revisited: weak convergence of probability measures *Ann. Probab.* **8** 1177–82
[10] Li X and Chen G R 2003 Synchronization and desynchronization of complex dynamical networks: an engineering viewpoint *IEEE Circuits Syst.* I **50** 1381–90
[11] Li X, Wang X F and Chen G R 2004 Pinning a complex dynamical network to its equilibrium *IEEE Trans. Circuits Syst.* I **51** 2074–87
[12] Lu X B, Wang X F, Li X and Fang J Q 2006 Synchronization in weighted complex networks: heterogeneity and synchronizability *Physica* A **370** 381–9
[13] Mohammed S-E A and Scheutzow M K R 1999 The stable manifold theorem for stochastic differential equations *Ann. Probab.* **27** 615–52
[14] Pikovsky A, Rosenblum M and Kurths J 2001 *Synchronization, a Universal Concept in Nonlinear Science* (Cambridge: Cambridge University Press)
[15] Qian M P and Wang D 2000 On a system of hyperstable frequency locking persistence under white noise *Ergod. Theo. Dyn. Syst.* **20** 547–55
[16] Qian M and Zhang F X 2005 Non-equilibrium of a general stochastic system of coupled oscillators: entropy production rate and rotation numbers *Ergod. Theo. Dyn. Syst.* **25** 1633–41

[17] Wang X F and Chen G R 2002 Synchronization in scale-free dynamical networks: robustness and fragility *IEEE Trans. Circuits Syst.* I **49** 54–62
[18] Wang X F and Chen G R 2002 Synchronization in small-world dynamical networks *Int. J. Bifurcation Chaos* **12** 187–92
[19] Wu C W 2005 Synchronization in networks of nonlinear dynamical systems coupled via a directed graph *Nonlinearity* **18** 1057–64
[20] Wu C W and Chua L O 1995 Synchronization in an array of linearly coupled dynamical systems *IEEE Trans. Circuits Syst.* I **42** 430–47
[21] Stroock D W 1999 *Probability Theory: an Analytic View* (Cambridge: Cambridge University Press)